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Transverse instability of surface solitary waves

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The linear stability of finite-amplitude surface solitary waves with respect to long-wavelength transverse perturbations is examined by asymptotic analysis for small wavenumbers of perturbations. The sufficient condition for the transverse instability is explicitly derived, and it is found that there exist transversely unstable surface solitary waves whose amplitude-to-depth ratio is greater than 0.713. This critical ratio is well below that for the one-dimensional instability (= 0.781) obtained by Tanaka (*Phys. Fluids*, 1986, vol. 29, p. 650).

1. Introduction

We here examine the transverse stability of surface solitary waves analytically. Transverse stability means a stability to perturbations that depend not only on the main wave's travelling direction, but also on its transverse direction. In contrast, one-dimensional stability is a stability to perturbations that depend only on the main wave's travelling direction. In the present study, we reveal the existence of surface solitary waves that are one-dimensionally stable, but are transversely unstable.

The one-dimensional stability of small-amplitude solitary waves was first examined by Jeffrey & Kakutani (1970) and later by Benjamin (1972) in the framework of the celebrated Korteweg–de Vries (KdV) equation. It was then found that these waves (small-amplitude solitary waves) are one-dimensionally stable. The study of transverse stability was initiated by Kadomtsev & Petviashvili (1970). They derived the Kadomtsev–Petviashvili (KP) equation, which is a two-dimensional extension of the KdV equation, and, on the basis of this equation, made a linear stability analysis of solitary waves with respect to long-wavelength transverse perturbations. The complete linear stability analysis with no restriction on the wavelength of perturbations was later conducted independently by Zakharov (1975), Kuznetsov, Spector & Fal'kovich (1984), Alexander, Pego & Sachs (1997), and Allen & Rowlands (1997). Then it was found that, in a medium with negative dispersion where the effects of surface tension are negligible, small-amplitude solitary waves are stable to transverse perturbations also.

The stability of finite-amplitude solitary waves was first examined by Tanaka (1986), who investigated the one-dimensional stability of surface solitary waves numerically. He found that an exchange of stability occurs at the first stationary point of the total energy. The critical amplitude-to-depth ratio is 0.781. Tanaka *et al.* (1987) also conducted numerical simulation of the time development of perturbed solitary waves, and the growth rate of sufficiently small perturbations agreed well with that of the linear stability analysis. The more precise calculation of the eigenvalue problem for linear perturbations was made by Longuet-Higgins & Tanaka (1997). They found that, if the amplitude-to-depth ratio is larger than 0.781, or the first stationary point of the total energy, the number of growing perturbation modes increases every time the solitary wave solutions pass through the stationary points of their total energy. In this way, one-dimensional stability of the surface solitary waves was fully investigated.

In contrast, it is only recently that the study on the transverse stability of finite-amplitude solitary waves was initiated. Bridges (2001) examined the linear stability of surface solitary waves to long-wavelength transverse perturbations. According to his analysis, the surface solitary waves are at the neutral stability to transverse perturbations of small wavenumbers. However, his analysis is based on the leading-order effects of the small wavenumbers only. The higher-order effects will determine the stability.

In the present study, these higher-order effects are investigated by the usual asymptotic analysis. It is revealed that there exist transversely unstable surface solitary waves. The critical amplitude-to-depth ratio obtained for the transverse stability is 0.713, which is well below that for the one-dimensional stability (= 0.781).

In § 2, we formulate the set of basic equations for linear transverse perturbations, which is reduced to the eigenvalue problem. In § 3, this eigenvalue problem is solved under the condition that the transverse wavenumbers of perturbations are small. Then the sufficient condition for transverse instability is derived, and this gives the critical amplitude-to-depth ratio of 0.713. Finally in § 4, some concluding remarks are given.

2. Basic equations

Consider the irrotational flow of an incompressible ideal fluid of undisturbed depth D with free surface under uniform acceleration due to gravity g. The effects of surface tension are neglected. In what follows, all variables are non-dimensionalized using g and D. Introducing the Cartesian coordinate $x - y - z$ with z pointed vertically upward and its origin placed on the undisturbed free surface, we obtain the following set of non-dimensional governing equations for the flow:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad [-1 < z < \eta(x, y, t)], \tag{2.1}$$

with the boundary conditions:

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{at } z = -1, \tag{2.2}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \eta}{\partial y} = \frac{\partial \phi}{\partial z} \quad \text{at } z = \eta(x, y, t), \tag{2.3}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] + \eta = f(t) \quad \text{at } z = \eta(x, y, t), \tag{2.4}$$

where ϕ is the velocity potential, η is the vertical displacement of the free surface, t is the time, and $f(t)$ is the value of terms on the left-hand side of (2.4) evaluated as $x \to \infty$.

Let us seek a solution of (2.1)–(2.4) in the following form:

$$\phi = -vx + \Phi_s(x, z), \tag{2.5a}$$

$$\eta = \eta_s(x), \tag{2.5b}$$

where $\partial \Phi_s/\partial x$, $\partial \Phi_s/\partial z$, and η_s approach zero as $x \to \pm\infty$, and v is a positive real parameter. This solution represents a steady propagation of localized wave against a uniform stream of constant speed v in the negative x-direction. We call this solution a solitary wave solution. Existence of such a solitary wave solution has already been confirmed numerically (Tanaka 1986; Longuet-Higgins & Tanaka 1997). The solution is known to exist if $1 < v \leqslant 1.294$, and has the property that $\partial\Phi_s/\partial x$, $\partial\Phi_s/\partial z$ and η_s decay exponentially as $x \to \pm\infty$.

Now we make a linear stability analysis of the solitary wave solution (2.5*a,b*) on the basis of (2.1)–(2.4). The following form is assumed for the solution of (2.1)–(2.4):

$$\phi = -vx + \Phi_s + \hat{\phi}(x, z)\exp(\lambda t + \mathrm{i}\,\varepsilon y), \tag{2.6a}$$

$$\eta = \eta_s + \hat{\eta}(x)\exp(\lambda t + \mathrm{i}\,\varepsilon y), \tag{2.6b}$$

where ε is a given real constant and λ is a real or a complex constant which is determined by solving the set of equations for $\hat{\phi}$ and $\hat{\eta}$ derived from (2.1)–(2.4). Note that the solitary wave solution is unstable if there exists a localized solution whose λ has a positive real part. In prior studies (Tanaka 1986; Longuet-Higgins & Tanaka 1997), the stability with respect to perturbations that have no dependence on y (the case of $\varepsilon = 0$), or the one-dimensional stability was examined numerically. According to their studies, one-dimensionally stable surface solitary waves satisfy

$$\frac{\mathrm{d}E}{\mathrm{d}v} > 0, \tag{2.7}$$

where E is the total energy of the solitary wave defined by

$$E = \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-1}^{\eta_s}\left[\left(\frac{\partial\Phi_s}{\partial x}\right)^2 + \left(\frac{\partial\Phi_s}{\partial z}\right)^2\right]\mathrm{d}z + \int_{-\infty}^{\infty}\tfrac{1}{2}\eta_s^2\,\mathrm{d}x. \tag{2.8}$$

Here we investigate the transverse stability, or the stability with respect to perturbations that depend not only on the x-direction, but also on the y-direction, of one-dimensionally stable solitary waves. Substituting (2.6*a,b*) into (2.1)–(2.4), linearizing with respect to $(\hat{\phi}, \hat{\eta})$, and imposing decaying condition as $x \to \pm\infty$, we obtain the following set of equations for $(\hat{\phi}, \hat{\eta})$:

$$\Delta\hat{\phi} = \varepsilon^2\hat{\phi}, \quad [-1 < z < \eta_s(x)], \tag{2.9}$$

with boundary conditions:

$$\frac{\partial\hat{\phi}}{\partial z} = 0 \quad \text{at } z = -1, \tag{2.10}$$

$$\boldsymbol{L}_1[\hat{\phi}, \hat{\eta}] = -\lambda\hat{\eta} \quad \text{at } z = \eta_s(x), \tag{2.11}$$

$$\boldsymbol{L}_2[\hat{\phi}, \hat{\eta}] = -\lambda\hat{\phi} \quad \text{at } z = \eta_s(x), \tag{2.12}$$

$$\hat{\phi}(x, z) \to 0, \quad \hat{\eta}(x) \to 0 \quad \text{as } x \to \pm\infty, \tag{2.13}$$

where Δ, $\boldsymbol{L}_1$, and $\boldsymbol{L}_2$ are the linear operators defined by

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}, \tag{2.14}$$

$$\boldsymbol{L}_1[\hat{\phi}, \hat{\eta}] = \left(-\frac{\partial}{\partial z} + \frac{\mathrm{d}\eta_s}{\mathrm{d}x}\frac{\partial}{\partial x}\right)\hat{\phi} + \left[\frac{\partial^2\Phi_s}{\partial x^2} + \frac{\partial^2\Phi_s}{\partial x\partial z}\frac{\mathrm{d}\eta_s}{\mathrm{d}x} + \left(-v + \frac{\partial\Phi_s}{\partial x}\right)\frac{\mathrm{d}}{\mathrm{d}x}\right]\hat{\eta}, \tag{2.15}$$

$$\boldsymbol{L}_2[\hat{\phi}, \hat{\eta}] = \left[\left(-v + \frac{\partial\Phi_s}{\partial x}\right)\frac{\partial}{\partial x} + \frac{\partial\Phi_s}{\partial z}\frac{\partial}{\partial z}\right]\hat{\phi} + \left[\left(-v + \frac{\partial\Phi_s}{\partial x}\right)\frac{\partial^2\Phi_s}{\partial x\partial z} + \frac{\partial\Phi_s}{\partial z}\frac{\partial^2\Phi_s}{\partial z^2} + 1\right]\hat{\eta}. \tag{2.16}$$

Equations (2.9)–(2.13) constitute the eigenvalue problem for $(\hat{\phi}, \hat{\eta})$ whose eigenvalue is λ. In the next section we solve this eigenvalue problem (2.9)–(2.13) for small ε in order to study the stability of solitary waves to long-wavelength transverse perturbations.

3. Stability to long-wavelength transverse perturbations

We seek an asymptotic solution of (2.9)–(2.13) for small ε under the condition that the solitary wave solution (Φ_s, η_s) satisfies (2.7). At leading order in ε, the term on the right-hand side of (2.9) can be ignored, and the eigenvalue problem (2.9)–(2.13) has the following leading-order steady solution:

$$\hat{\phi} = \hat{\phi}_{C0} \equiv \frac{\partial \Phi_s}{\partial x}, \quad \hat{\eta} = \hat{\eta}_{C0} \equiv \frac{\mathrm{d}\eta_s}{\mathrm{d}x}, \quad \lambda = 0. \tag{3.1}$$

Equation (3.1) will be subject to a slow time development if the term on the right-hand side of (2.9), or the term of $O(\varepsilon^2)$ is recovered. In what follows, we investigate the asymptotic behaviour for small ε under the assumption that its slow time development is described by the two different time scales of $O(\varepsilon^{-1})$ and $O(\varepsilon^{-2})$. So we express λ as

$$\lambda = \varepsilon\lambda_1 + \varepsilon^2\lambda_2. \tag{3.2}$$

The validity of this estimate is confirmed if the following analysis is consistent.

3.1. *Core solution*

First, we look for the solution of (2.9)–(2.12) with an appreciable variation in x and z of the order of unity ($\partial\hat{\phi}/\partial x = O(\hat{\phi})$, $\partial\hat{\phi}/\partial z = O(\hat{\phi})$ and $\mathrm{d}\hat{\eta}/\mathrm{d}x = O(\hat{\eta})$) in the power series of ε:

$$\hat{\phi}_C = \hat{\phi}_{C0} + \varepsilon\hat{\phi}_{C1} + \varepsilon^2\hat{\phi}_{C2} + \cdots, \tag{3.3a}$$

$$\hat{\eta}_C = \hat{\eta}_{C0} + \varepsilon\hat{\eta}_{C1} + \varepsilon^2\hat{\eta}_{C2} + \cdots, \tag{3.3b}$$

where the subscript C is attached to indicate the type of solution (core solution).

Substituting the series (3.2) and (3.3*a*, *b*) into (2.9)–(2.12) and arranging the same-order terms in ε, we obtain a series of equations for $(\hat{\phi}_{Cn}, \hat{\eta}_{Cn})$ $(n = 1, 2, \cdots)$:

$$\Delta\hat{\phi}_{Cn} = \hat{\phi}_{Cn-2} \quad [-1 < z < \eta_s(x)], \tag{3.4}$$

$$\frac{\partial\hat{\phi}_{Cn}}{\partial z} = 0 \quad \text{at } z = -1, \tag{3.5}$$

$$\boldsymbol{L}_1[\hat{\phi}_{Cn}, \hat{\eta}_{Cn}] = G_n \equiv -\lambda_1\hat{\eta}_{Cn-1} - \lambda_2\hat{\eta}_{Cn-2} \quad \text{at } z = \eta_s(x), \tag{3.6}$$

$$\boldsymbol{L}_2[\hat{\phi}_{Cn}, \hat{\eta}_{Cn}] = H_n \equiv -\lambda_1\hat{\phi}_{Cn-1} - \lambda_2\hat{\phi}_{Cn-2} \quad \text{at } z = \eta_s(x), \tag{3.7}$$

where Δ, $\boldsymbol{L}_1$, and $\boldsymbol{L}_2$ are given by (2.14)–(2.16) and $\hat{\phi}_{C-1} = \hat{\eta}_{C-1} = 0$. Now consider the linear inhomogeneous equations (3.4)–(3.7) whose homogeneous parts have a non-trivial solution $(\hat{\phi}_{C0}, \hat{\eta}_{C0})$. The adjoint of the above homogeneous parts, which are given only by reversing the sign of terms including $\hat{\eta}_{Cn}$, have a solution $(\hat{\phi}_{C0}, -\hat{\eta}_{C0}) = (\partial\Phi_s/\partial x,\ -\mathrm{d}\eta_s/\mathrm{d}x)$ that decays exponentially as $x \to \pm\infty$. Therefore, for the inhomogeneous equations (3.4)–(3.7) to have a solution that does not diverge exponentially as $x \to \pm\infty$, their inhomogeneous terms on the right-hand sides of

(3.4)–(3.7) must satisfy the following relation (solvability condition):

$$\int_{-\infty}^{\infty} \mathrm{d}x \int_{-1}^{\eta_s} \frac{\partial \Phi_s}{\partial x} \hat{\phi}_{Cn-2}\, \mathrm{d}z + \int_{-\infty}^{\infty} \left[\frac{\partial \Phi_s}{\partial x} G_n - \frac{\mathrm{d}\eta_s}{\mathrm{d}x} H_n \right]_{z=\eta_s} \mathrm{d}x = 0, \tag{3.8}$$

where the value in the square bracket with subscript $z = \eta_s$ (or $[\ ]_{z=\eta_s}$) is evaluated at $z = \eta_s$.

At $n = 1$, we easily find that the solvability condition (3.8) is identically satisfied, and the solution of (3.4)–(3.7) at $n = 1$ is obtained as follows:

$$\hat{\phi}_{C1} = r_1 \frac{\partial \Phi_s}{\partial x} - \lambda_1 \frac{\partial \Phi_s}{\partial v}, \quad \hat{\eta}_{C1} = r_1 \frac{\mathrm{d}\eta_s}{\mathrm{d}x} - \lambda_1 \frac{\partial \eta_s}{\partial v}, \tag{3.9}$$

where r_1 is an arbitrary constant, and $\partial \Phi_s/\partial v$ and $\partial \eta_s/\partial v$ represent the derivatives of Φ_s and η_s with respect to v for fixed x and z. We can check that the solution (3.9) satisfies (3.4)–(3.7) at $n = 1$ by taking the derivative with respect to v of equations for the solitary wave solution (Φ_s, η_s). The solution (3.9), however, does not satisfy the boundary condition (2.13), since $\partial \Phi_s/\partial v$ in general does not become zero for both $x \to \pm\infty$. This situation continues to the higher orders. The reason for such inappropriateness is that we are not taking into account the balance between the terms including λ and those including $-v\partial/\partial x$ of (2.11) and (2.12). To achieve the balance of these terms, we need to introduce a shrunk coordinate with respect to x and seek a solution whose variation occurs slowly in x. This solution will be called a far-field solution. By matching the core solution (3.3*a*, *b*) with the far-field solution, the solution of (2.9)–(2.12) that satisfies the boundary condition (2.13) can be constructed. In this subsection we concentrate on obtaining the core solution putting aside the boundary condition (2.13), while § 3.2 considers the far-field solution.

At $n = 2$, the solvability condition (3.8) is

$$\frac{\lambda_1^2}{v} \frac{\mathrm{d}E}{\mathrm{d}v} = -\int_{-\infty}^{\infty} \mathrm{d}x \int_{-1}^{\eta_s} \left(\frac{\partial \Phi_s}{\partial x} \right)^2 \mathrm{d}z, \tag{3.10}$$

where E is defined by (2.8). Recalling that we have assumed that the solitary wave solution satisfies (2.7), we obtain

$$\lambda_1 = \pm \mathrm{i} \sqrt{\frac{v}{\mathrm{d}E/\mathrm{d}v} \int_{-\infty}^{\infty} \mathrm{d}x \int_{-1}^{\eta_s} \left(\frac{\partial \Phi_s}{\partial x} \right)^2 \mathrm{d}z}. \tag{3.11}$$

This is the first-order criterion derived by Bridges (2001). However, the real part of λ_1 is zero so that the stability of the solitary wave solution is not determined at this order. To know the stability, we must proceed to the next order.

At $n = 3$, the solvability condition (3.8) becomes

$$\frac{2\lambda_1\lambda_2}{v} \frac{\mathrm{d}E}{\mathrm{d}v} = [\hat{\phi}_{C1}\hat{u}_{C2}]_{x\to-\infty} - [\hat{\phi}_{C1}\hat{u}_{C2}]_{x\to\infty}, \tag{3.12}$$

where

$$\hat{u}_{C2} \equiv \int_{-1}^{\eta_s} \frac{\partial \hat{\phi}_{C2}}{\partial x}\, \mathrm{d}z + \left(-v + \frac{\partial \Phi_s}{\partial x} \right) \hat{\eta}_{C2} \tag{3.13a}$$

$$= [\hat{u}_{C2}]_{x\to\infty} - (r_1\lambda_1 + \lambda_2)\eta_s + \int_{\infty}^{x} \left(\int_{-1}^{\eta_s} \frac{\partial \Phi_s}{\partial x}\, \mathrm{d}z + \lambda_1^2 \frac{\partial \eta_s}{\partial v} \right) \mathrm{d}x' \tag{3.13b}$$

and the values in the square bracket with subscript $x \to \pm\infty$ (or $[\ ]_{x\to\pm\infty}$) are evaluated as $x \to \pm\infty$. The key to deriving (3.12) is to note that $-\lambda_1 \partial \Phi_s/\partial x$ and

$-\lambda_1\,\mathrm{d}\eta_s/\mathrm{d}x$ included in the solvability condition (3.8) at $n=3$ are, respectively, equal to the inhomogeneous terms of (3.6) and (3.7) at $n=1$, and replace them with the corresponding left-hand-side terms, or $\boldsymbol{L}_1[\hat{\phi}_{C1},\hat{\eta}_{C1}]$ and $\boldsymbol{L}_2[\hat{\phi}_{C1},\hat{\eta}_{C1}]$. Then, using the integration by parts and integration of (3.4) at $n=2$ multiplied by $\hat{\phi}_{C1}$ over the whole fluid domain, we obtain (3.12). We can derive (3.13*b*) by integrating (3.4) at $n=2$ with respect to x, z, and using (3.5) and (3.6) at $n=2$. $[\hat{\phi}_{C1}]_{x\to-\infty}$ and $[\hat{u}_{C2}]_{x\to-\infty}$ appearing on the right-hand side of (3.12) are related to the corresponding values as $x\to\infty$ by

$$[\hat{\phi}_{C1}]_{x\to-\infty}=[\hat{\phi}_{C1}]_{x\to\infty}+\lambda_1\frac{\mathrm{d}}{\mathrm{d}v}\left(I-\frac{2T}{v}\right), \tag{3.14a}$$

$$[\hat{u}_{C2}]_{x\to-\infty}=[\hat{u}_{C2}]_{x\to\infty}-I-\lambda_1^2\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{I}{v}\right), \tag{3.14b}$$

where I and T are, respectively, the impulse and the kinetic energy of the solitary wave:

$$I=\int_{-\infty}^{\infty}\mathrm{d}x\int_{-1}^{\eta_s}\frac{\partial\Phi_s}{\partial x}\,\mathrm{d}z=v\int_{-\infty}^{\infty}\eta_s\,\mathrm{d}x, \tag{3.15}$$

$$T=\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-1}^{\eta_s}\left[\left(\frac{\partial\Phi_s}{\partial x}\right)^2+\left(\frac{\partial\Phi_s}{\partial z}\right)^2\right]\mathrm{d}z. \tag{3.16}$$

The condition (3.12) will generally give the non-zero real part of λ_2. Thus, the core solution is obtained up to the orders that can determine the stability of the solitary wave. The specific values of $[\hat{\phi}_{C1}]_{x\to\infty}$ and $[\hat{u}_{C2}]_{x\to\infty}$ necessary for determining the value of λ_2, are given after matching the core solution with the far-field solution, which will be shown in § 3.3. In § 3.2, the far-field solution is investigated.

3.2. *Far-field solution*

We seek a solution considering the balance between the terms with λ and those with $-v\partial/\partial x$ of (2.11) and (2.12). To this end, in accordance with the introduction of $\varepsilon\lambda_1$ and $\varepsilon^2\lambda_2$ in (3.2), we here introduce two shrunk coordinates with respect to x:

$$X_1=\varepsilon x,\quad X_2=\varepsilon^2 x. \tag{3.17}$$

We then look for the solution of (2.9)–(2.12) whose appreciable variation occurs in X_1, X_2, and z of the order of unity, ($\partial\hat{h}/\partial X_1=O(\hat{h})$, $\partial\hat{h}/\partial X_2=O(\hat{h})$ and $\partial\hat{\phi}/\partial z=O(\hat{\phi})$, where $\hat{h}$ represents $\hat{\phi}$ and $\hat{\eta}$), in the following power series of ε:

$$\hat{\phi}_F=\varepsilon\hat{\phi}_{F1}(X_1,X_2,z)+\varepsilon^2\hat{\phi}_{F2}(X_1,X_2,z)+\cdots, \tag{3.18a}$$

$$\hat{\eta}_F=\varepsilon^2\hat{\eta}_{F2}(X_1,X_2)+\varepsilon^3\hat{\eta}_{F3}(X_1,X_2)+\cdots, \tag{3.18b}$$

where the subscript F is attached to show the type of solution (far-field solution). The series of (3.18*a,b*) start from $O(\varepsilon)$ and $O(\varepsilon^2)$ for $\hat{\phi}_F$ and $\hat{\eta}_F$, respectively, in accordance with the core solution not being able to satisfy the boundary condition as $x\to\pm\infty$ from these orders (see (3.14*a,b*)).

Substituting (3.17) and (3.18*a, b*) into (2.9)–(2.12), and arranging the same-order terms in ε, we obtain a series of equations for $\hat{\phi}_{Fn}$ ($n=1,2,\ldots$):

$$\frac{\partial^2\hat{\phi}_{Fn}}{\partial z^2}=I_n\equiv\hat{\phi}_{Fn-2}-\frac{\partial^2\hat{\phi}_{Fn-2}}{\partial X_1^2}-2\frac{\partial^2\hat{\phi}_{Fn-3}}{\partial X_1\partial X_2}-\frac{\partial^2\hat{\phi}_{Fn-4}}{\partial X_2^2}, \tag{3.19}$$

with the boundary conditions:

$$\frac{\partial \hat{\phi}_{Fn}}{\partial z} = 0 \quad \text{at } z = -1, \tag{3.20}$$

$$\frac{\partial \hat{\phi}_{Fn}}{\partial z} = J_n \equiv \lambda_1 \hat{\eta}_{Fn-1} - v\frac{\partial \hat{\eta}_{Fn-1}}{\partial X_1} + \lambda_2 \hat{\eta}_{Fn-2} - v\frac{\partial \hat{\eta}_{Fn-2}}{\partial X_2} \quad \text{at } z = 0, \tag{3.21}$$

where $\hat{\phi}_{Fm}$ ($m \leqslant 0$) and $\hat{\eta}_{Fm}$ ($m \leqslant 1$) appearing on the right-hand sides of (3.19) and (3.21) are zero. $\hat{\eta}_{Fn}$ ($n = 2, 3, \ldots$) can be obtained by the following equation in terms of $\hat{\phi}_{Fn-1}$ and $\hat{\phi}_{Fn-2}$:

$$\hat{\eta}_{Fn} = \left[-\lambda_1 \hat{\phi}_{Fn-1} + v\frac{\partial \hat{\phi}_{Fn-1}}{\partial X_1} - \lambda_2 \hat{\phi}_{Fn-2} + v\frac{\partial \hat{\phi}_{Fn-2}}{\partial X_2}\right]_{z=0}. \tag{3.22}$$

Equations (3.19)–(3.21) (with the subsidiary equation (3.22)) constitute the system of equations for $\hat{\phi}_{Fn}$ ($n = 1, 2, \ldots$). At $n = 1$ and 2, they are homogeneous and have the following solutions:

$$\hat{\phi}_{F1} = \hat{\phi}_{F1}(X_1, X_2), \quad \hat{\phi}_{F2} = \hat{\phi}_{F2}(X_1, X_2). \tag{3.23}$$

At $n = 3, 4, \ldots$, they are inhomogeneous. For these inhomogeneous equations to have a solution, their inhomogeneous terms I_n and J_n must satisfy the following relation (solvability condition):

$$\int_{-1}^{0} I_n \, \mathrm{d}z - J_n = 0. \tag{3.24}$$

Specifically, at $n = 3$, the solvability condition (3.24) is

$$(v^2 - 1)\frac{\partial^2 \hat{\phi}_{F1}}{\partial X_1^2} - 2v\lambda_1 \frac{\partial \hat{\phi}_{F1}}{\partial X_1} + (\lambda_1^2 + 1)\hat{\phi}_{F1} = 0. \tag{3.25}$$

This equation determines the dependence of $\hat{\phi}_{F1}$ on X_1 as

$$\hat{\phi}_{F1} = q_1 \exp(kX_1) + \bar{q}_1 \exp(\bar{k}X_1), \tag{3.26}$$

where q_1 and $\bar{q}_1$ are undetermined functions of X_2, and

$$k = \frac{v + a}{v^2 - 1}\lambda_1, \quad \bar{k} = \frac{v - a}{v^2 - 1}\lambda_1, \tag{3.27}$$

are both pure imaginary constants with

$$a = \sqrt{1 + \frac{v^2 - 1}{|\lambda_1|^2}} \quad (1 < v \leqslant 1.294). \tag{3.28}$$

At $n = 4$, the solvability condition (3.24) is

$$(v^2 - 1)\frac{\partial^2 \hat{\phi}_{F2}}{\partial X_1^2} - 2v\lambda_1 \frac{\partial \hat{\phi}_{F2}}{\partial X_1} + (\lambda_1^2 + 1)\hat{\phi}_{F2}$$
$$= -2\left\{\left[(v^2 - 1)\frac{\partial}{\partial X_1} - v\lambda_1\right]\frac{\partial}{\partial X_2} - \lambda_2\left(v\frac{\partial}{\partial X_1} - \lambda_1\right)\right\}\hat{\phi}_{F1}. \tag{3.29}$$

For this inhomogeneous equation (3.29) to have a solution that does not diverge with respect to X_1, its inhomogeneous term on the right-hand side must be zero. From this condition, the dependence of $\hat{\phi}_{F1}$ on X_2, or the functional form of q_1 and $\bar{q}_1$ is determined as

$$q_1 = c_{1\pm} \exp\left[\frac{va+1}{(v^2-1)a}\lambda_2 X_2\right], \quad \bar{q}_1 = \bar{c}_{1\pm} \exp\left[\frac{va-1}{(v^2-1)a}\lambda_2 X_2\right], \tag{3.30}$$

where c_{1+} and $\bar{c}_{1+}$ are undetermined constants for $X_1, X_2 > 0$, and c_{1-} and $\bar{c}_{1-}$ are those for $X_1, X_2 < 0$. Thus, we have obtained the leading-order far-field solution given by (3.26) with (3.30) and (3.22) at $n = 2$.

3.3. *Matching of the core solution and the far-field solution*

Here, we carry out the matching of the core solution $(\hat{\phi}_C, \hat{\eta}_C)$ and the far-field solution $(\hat{\phi}_F, \hat{\eta}_F)$. To this end, the far-field solution $(\hat{\phi}_{Fn}, \hat{\eta}_{Fn})$ in the core region is reshuffled by expanding in the power series of X_1 (or εx) and X_2 (or $\varepsilon^2 x$):

$$\hat{h}_{Fn} = (\hat{h}_{Fn})_0 + \varepsilon x \left(\frac{\partial \hat{h}_{Fn}}{\partial X_1}\right)_0 + \varepsilon^2 \left[\frac{x^2}{2}\left(\frac{\partial^2 \hat{h}_{Fn}}{\partial X_1^2}\right)_0 + x\left(\frac{\partial \hat{h}_{Fn}}{\partial X_2}\right)_0\right] + \cdots, \tag{3.31}$$

where $\hat{h}$ represents $(\hat{\phi}, \hat{\eta})$, and the quantities in the parentheses with subscript 0, or $(\cdots)_0$, are evaluated at $X_1 = X_2 = 0$. Then, we collect the same-order terms in ε and obtain the reordered form [say, $(\hat{\phi}^*_{Fn}, \hat{\eta}^*_{Fn})$] of $(\hat{\phi}_{Fn}, \hat{\eta}_{Fn})$. After this reordering, we compare the forms of the two solutions $(\hat{\phi}_{Cn}, \hat{\eta}_{Cn})$ and $(\hat{\phi}^*_{Fn}, \hat{\eta}^*_{Fn})$ at each n and carry out their matching from $n = 1$. The matching is accomplished if the conditions:

$$\hat{\phi}_{Cn} \sim \hat{\phi}^*_{Fn}, \quad \hat{\eta}_{Cn} \sim \hat{\eta}^*_{Fn} \quad \text{as } x \to \pm\infty, \tag{3.32}$$

are satisfied with differences being smaller than any inverse power of x.

At order ε, since $\hat{\phi}^*_{F1} = (\hat{\phi}_{F1})_0$, the matching conditions (3.32) are, from (3.26) with (3.30),

$$[\hat{\phi}_{C1}]_{x\to\pm\infty} = c_{1\pm} + \bar{c}_{1\pm}. \tag{3.33}$$

At order ε^2, since $\hat{\phi}^*_{F2} = (\hat{\phi}_{F2})_0 + x(\partial\hat{\phi}_{F1}/\partial X_1)_0$, the matching conditions for $\hat{\phi}$ are composed of two different kinds of terms, i.e. those independent of x and those proportional to x. The latter relations contribute to the determination of the unknown constants at this order. It is convenient to represent these relations in terms of $\hat{u}_{C2}$ defined by (3.13*a*). That is,

$$\hat{u}_{C2} \sim \int_{-1}^{0} \frac{\partial \hat{\phi}^*_{F2}}{\partial x}\,\mathrm{d}z - v\hat{\eta}^*_{F2} = \left(\frac{\partial \hat{\phi}_{F1}}{\partial X_1}\right)_0 - v(\hat{\eta}_{F2})_0. \tag{3.34}$$

Substituting (3.22) at $n = 2$ and (3.26) with (3.30) into (3.34), we obtain

$$[\hat{u}_{C2}]_{x\to\pm\infty} = a\lambda_1(\bar{c}_{1\pm} - c_{1\pm}), \tag{3.35}$$

where a is given by (3.28). The matching conditions for $\hat{\eta}$ are automatically satisfied if (3.33) and (3.35) are satisfied.

Moreover, from the boundary condition (2.13), we obtain, using the fact that the sign of the exponents in (3.30) is the same as that of λ_2,

$$c_{1+} = \bar{c}_{1+} = 0 \quad \text{when } \mathrm{Re}[\lambda_2] > 0, \tag{3.36a}$$

$$c_{1-} = \bar{c}_{1-} = 0 \quad \text{when } \mathrm{Re}[\lambda_2] < 0. \tag{3.36b}$$

The six undetermined constants $[\hat{\phi}_{C1}]_{x\to\infty}$, $[\hat{u}_{C2}]_{x\to\infty}$, c_{1+}, $\bar{c}_{1+}$, c_{1-} and $\bar{c}_{1-}$ (note that $[\hat{\phi}_{C1}]_{x\to-\infty}$ and $[\hat{u}_{C2}]_{x\to-\infty}$ are given by (3.14*a, b*)) are determined by the six equations

(3.33), (3.35) and (3.36*a*) or (3.36*b*). They are easy to solve, and we obtain

$$[\hat{\phi}_{C1}]_{x\to\infty} = 0, \tag{3.37a}$$

$$[\hat{u}_{C2}]_{x\to\infty} = 0, \tag{3.37b}$$

$$c_{1+} = \bar{c}_{1+} = 0, \tag{3.37c}$$

$$c_{1-} = \tfrac{1}{2}\lambda_1 \frac{\mathrm{d}}{\mathrm{d}v}\left(I - \frac{2T}{v}\right) + \frac{1}{2a\lambda_1}\left[I + {\lambda_1}^2 \frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{I}{v}\right)\right], \tag{3.37d}$$

$$\bar{c}_{1-} = \tfrac{1}{2}\lambda_1 \frac{\mathrm{d}}{\mathrm{d}v}\left(I - \frac{2T}{v}\right) - \frac{1}{2a\lambda_1}\left[I + {\lambda_1}^2 \frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{I}{v}\right)\right], \tag{3.37e}$$

when $\mathrm{Re}[\lambda_2] > 0$, and

$$[\hat{\phi}_{C1}]_{x\to\infty} = -\lambda_1 \frac{\mathrm{d}}{\mathrm{d}v}\left(I - \frac{2T}{v}\right), \tag{3.38a}$$

$$[\hat{u}_{C2}]_{x\to\infty} = I + \lambda_1^2 \frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{I}{v}\right), \tag{3.38b}$$

$$c_{1+} = -\tfrac{1}{2}\lambda_1 \frac{\mathrm{d}}{\mathrm{d}v}\left(I - \frac{2T}{v}\right) - \frac{1}{2a\lambda_1}\left[I + {\lambda_1}^2 \frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{I}{v}\right)\right], \tag{3.38c}$$

$$\bar{c}_{1+} = -\tfrac{1}{2}\lambda_1 \frac{\mathrm{d}}{\mathrm{d}v}\left(I - \frac{2T}{v}\right) + \frac{1}{2a\lambda_1}\left[I + {\lambda_1}^2 \frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{I}{v}\right)\right], \tag{3.38d}$$

$$c_{1-} = \bar{c}_{1-} = 0, \tag{3.38e}$$

when $\mathrm{Re}[\lambda_2] < 0$. Substituting (3.14) with (3.37*a*, *b*) or (3.38*a*, *b*) into the solvability condition (3.12), we obtain the following solution for λ_2:

$$\lambda_2 = \begin{cases} \pm Q & \text{if } Q < 0, \\ \text{no solution} & \text{if } Q > 0, \end{cases} \tag{3.39}$$

where

$$Q = \tfrac{1}{2}v\left[I + \lambda_1^2 \frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{I}{v}\right)\right]\frac{\mathrm{d}}{\mathrm{d}E}\left(I - \frac{2T}{v}\right). \tag{3.40}$$

The corresponding eigenfunction for the unstable case $\lambda_2 = -Q$ when $Q < 0$ is given by (3.3) with (3.1) and (3.9) for the core solution and (3.18) with (3.26), (3.30) and (3.37*c*–*e*) for the far-field solution up to the first order in ε. As for the second-order solution, we used $\hat{u}_{C2}$ and $\hat{\eta}_{F2}$ in order to accomplish matching, and they are given by (3.13*b*) with (3.37*b*) and (3.22) at $n = 2$, respectively.

Thus, we have a solution of the eigenvalue problem (2.9)–(2.13) whose λ has the positive real part (or $\lambda_2 = -Q > 0$) when $Q < 0$. Now we can say that a sufficient condition for transverse instability is

$$Q < 0. \tag{3.41}$$

Let us apply the criterion (3.41) to the solitary wave solution, which is calculated by the numerical method described in Tanaka (1986). The function $I + \lambda_1^2\,\mathrm{d}(I/v)/\mathrm{d}v$ included in Q is always positive. That is, when $\mathrm{d}(I/v)/\mathrm{d}v < 0$, it is clearly positive since $I > 0$ from (3.15) and ${\lambda_1}^2 < 0$ from (3.11), and when $\mathrm{d}(I/v)/\mathrm{d}v > 0$, the smaller function $I - 2vT\,\mathrm{d}(I/v)/\mathrm{d}E$ is numerically found to be always positive if $\mathrm{d}E/\mathrm{d}v > 0$. In figure 1, therefore, $I - 2T/v$ is plotted as a function of E. If this function takes negative gradient for the one-dimensionally stable range of $\mathrm{d}E/\mathrm{d}v > 0$, the corresponding wave is transversely unstable. The circle in the figure represents the

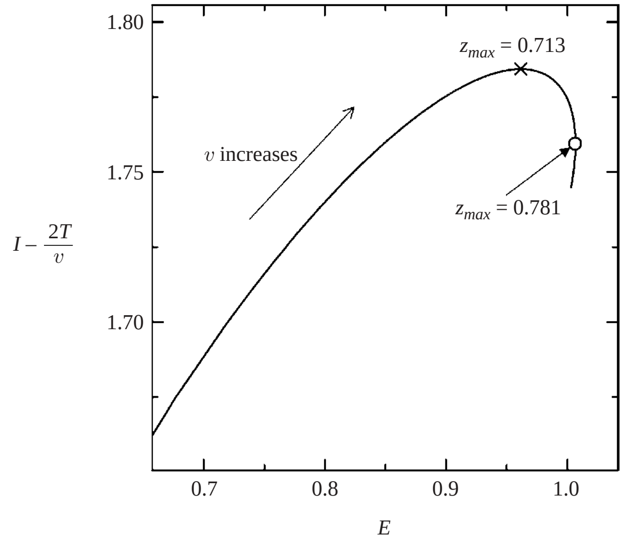

FIGURE 1. $I - 2T/v$ as a function of E. The wave speed v increases along the curve up to its endpoint shown in the figure. The cross represents the point of $\mathrm{d}(I - 2T/v)/\mathrm{d}v = 0$, or $Q = 0$, and the open circle represents that of $\mathrm{d}E/\mathrm{d}v = 0$. The corresponding amplitude-to-depth ratios z_{max} are also shown in the figure.

point of $\mathrm{d}E/\mathrm{d}v = 0$, or the amplitude-to-depth ratio (hereinafter denoted by z_{max}) of 0.781, and the solitary wave is one-dimensionally stable, or $\mathrm{d}E/\mathrm{d}v > 0$ for v smaller than that of the circle (or the upper branch from the circle in figure 1). We see that the function $I - 2T/v$ takes negative gradient from the cross to the circle. Therefore, there exist solitary waves that are one-dimensionally stable, but are transversely unstable in this range. The cross in figure 1 represents the critical point for the transverse stability, and its amplitude-to-depth ratio z_{max} is 0.713 ($v = 1.282$), which is well below that for the one-dimensional stability $z_{max} = 0.781$ ($v = 1.294$).

Thus, an existence of transversely unstable surface solitary waves for $0.713 < z_{max} < 0.781$ ($1.282 < v < 1.294$) has been revealed.

4. Concluding remarks

We have examined the linear stability of finite-amplitude surface solitary waves with respect to long-wavelength transverse perturbations. The criterion for the transverse instability (3.41) is explicitly derived, and it is found that there exist transversely unstable surface solitary waves whose amplitude-to-depth ratio is greater than 0.713, which is well below that for the one-dimensional critical instability 0.781.

The long-term evolution of these unstable waves under the influence of long-wavelength transverse perturbations is as follows. At the initial stage when the linear analysis is valid, they evolve according to the linear asymptotic analysis described in § 3. That is, the wave oscillates periodically both in time and transverse (y) direction (represented by λ_1; see (3.11)), while the amplitude of this oscillation grows exponentially, but at a fairly slow pace (represented by λ_2; see (3.39)). In the later stage, when nonlinear effects are not negligible, two distinctive forms of evolution are expected to occur. In one case, the instability leads to the collapse; in the other case, the continuous radiation of perturbation waves (represented by the far-field solution) decreases the energy and the amplitude of the solitary wave significantly so that it reduces to a stable solitary wave of smaller amplitude ($z_{max} < 0.713$). The possible occurrence of this latter case is due to a fairly small growth rate of perturbations. Either of these evolutions is expected to take place at the nonlinear stage of instability; however, this problem needs further investigation.

Finally, we make remarks about the stability to finite-wavelength perturbations. Kataoka, Tsutahara & Negoro (2000) discovered that there exist KdV-type solitary wave solutions that are unstable to long-wavelength transverse perturbations in the framework of the generalized Kadomtsev–Petviashvili (GKP) equation with negative dispersion. The authors found that there is a short-wavelength cutoff to this transverse instability (Kataoka & Tsutahara 2004). That is, the solitary wave solutions are more stable to the shorter-wavelength perturbations in the framework of the GKP equation. The same tendency is also expected for the present surface-solitary-wave case, since the instability is of the same type in that perturbations grow fairly slowly while oscillating periodically. However, this speculation requires numerical examination to justify it.

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